Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Student Number

2023 **Year 12 Examination**

Mathematics Extension 1

Trial

Date 3rd August, 2023

Q	Marks
MC	/10
11	/16
12	/15
13	/16
14	/13
Total	/70

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using blue or black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

Total Marks: 70

Section I - 10 marks

- Allow about 15 minutes for this section

Section II - 60 marks

- Allow about 1 hour 45 minutes for this section

This question paper must not be removed from the examination room.

This assessment task constitutes 40% of the course.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section.

Use the multiple-choice sheet for Question 1–10.

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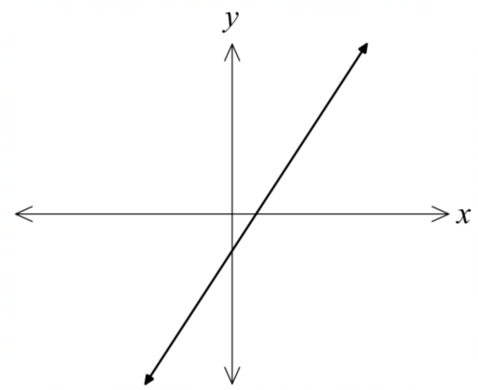
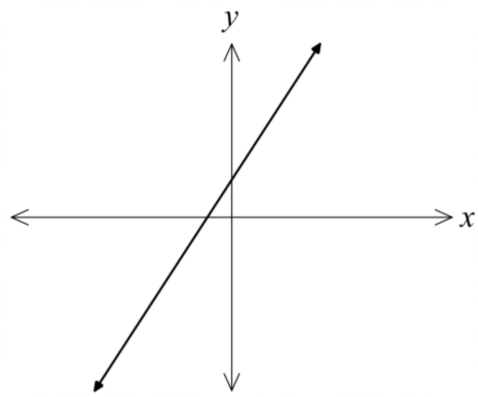
1 Which of the following graphs could represent the parametric equations given below?

$$\begin{cases} x = 2t + 1 \\ y = 4t + 1 \end{cases}$$

A.



B.



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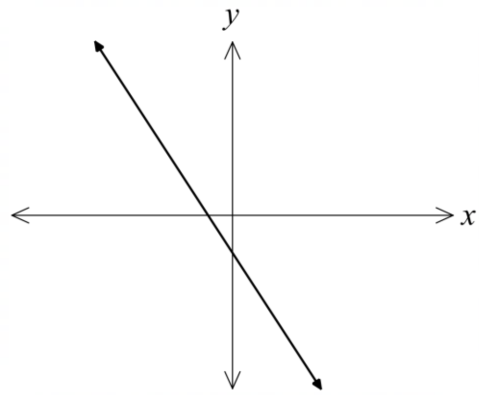
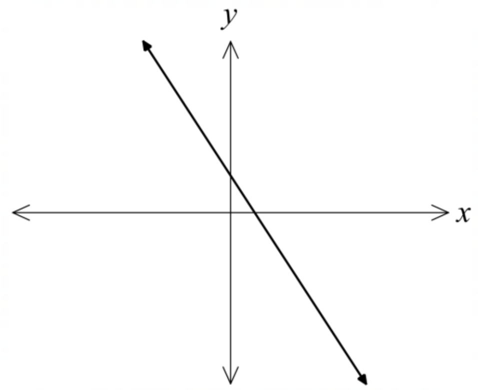


D.



2 When $P(x) = x^3 - 2x^2 + 6x + 7$ is divided by $(x - 3)$, the remainder is

A. -34

B. -7

C. 7

D. 34

3 Determine the equation that is the solution of

$$\frac{dy}{dx} = 6y^2x$$

A. $y = \frac{3}{c-x^2}$

B. $y = \frac{1}{c-3x^2}$

C. $y = \frac{1}{c+3x^2}$

D. $y = \frac{3}{c+x^2}$

4 A drawer contains 12 red socks and 12 blue socks. A person takes socks from the drawer in the dark. How many socks must they take to ensure they have at least one pair of blue socks?

A. 3

B. 5

C. 13

D. 14

5 The two vectors $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \sqrt{3}\underset{\sim}{i} + \underset{\sim}{j}$ are placed tip to tail. What is the angle between the two vectors?

A. $\frac{\pi}{12}$

B. $\frac{\pi}{3}$

C. $\frac{7\pi}{12}$

D. $\frac{11\pi}{12}$

6 A lottery has 30 numbers and 6 are drawn without replacement. A player of this game chooses 6 distinct numbers themselves.

The probability of matching at least 4 of the 6 numbers is given by:

A. 0.005

B. 0.007

C. 0.018

D. 0.046

7 The number of ways the letters in the word STATISTICS can be arranged to form different arrangements are

A. 25000

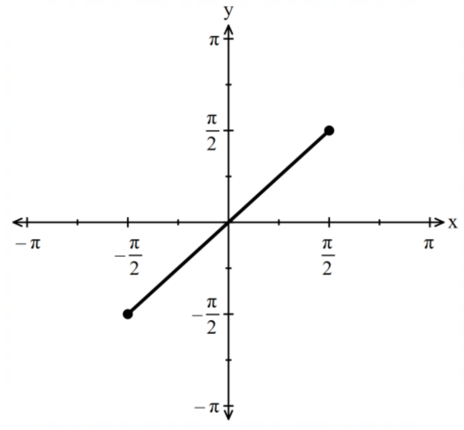
B. 25200

C. 50400

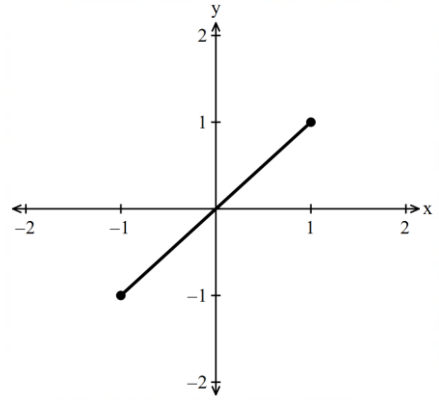
D. 100800

8 Which of the follow sketches represents $y = \cos^{-1}(\cos(x))$

A.



B.



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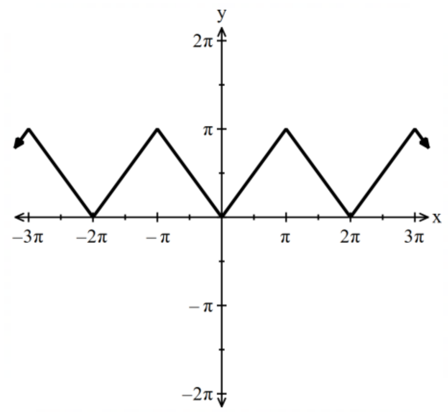
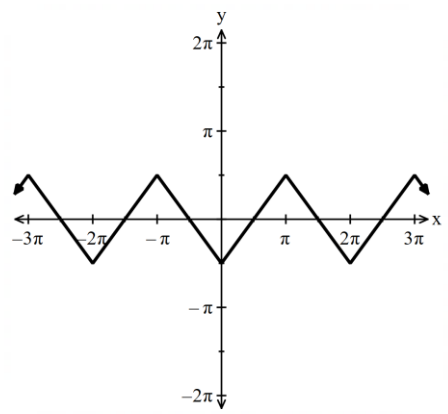


D.



9 Let $g(x) = \frac{ax+b}{cx+d}$ where $a, b, c\ and\ d$ are positive and $ad \neq bc$. Which of the following can not in the domain of the inverse function?

A. $\dfrac{a}{a}$

B. $\dfrac{a}{b}$

C. $\dfrac{a}{c}$

D. $\dfrac{a}{d}$

10 Consider two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$, where $\underset{\sim}{u} \cdot \underset{\sim}{v} < 0$.

It is known $\left| proj_{\underset{\sim}{u}}\underset{\sim}{v} \right| = k$.

Which of the following expression gives the value for $\left| proj_{\underset{\sim}{u}}\left(\underset{\sim}{u} + \underset{\sim}{v}\right) \right|$?

A. $\left|\underset{\sim}{u}\right| - k$

B. $\left|\underset{\sim}{u}\right| + k$

C. $\left|\underset{\sim}{u} + \underset{\sim}{v}\right| - k$

D. $\left|\underset{\sim}{u} + \underset{\sim}{v}\right| + k$

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Answer each question in a NEW booklet.

(a) If $\underset{\sim}{a} = -\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} - 2\underset{\sim}{j}$, find $\underset{\sim}{a} + \underset{\sim}{b}$. **1**

(b) Evaluate **2**

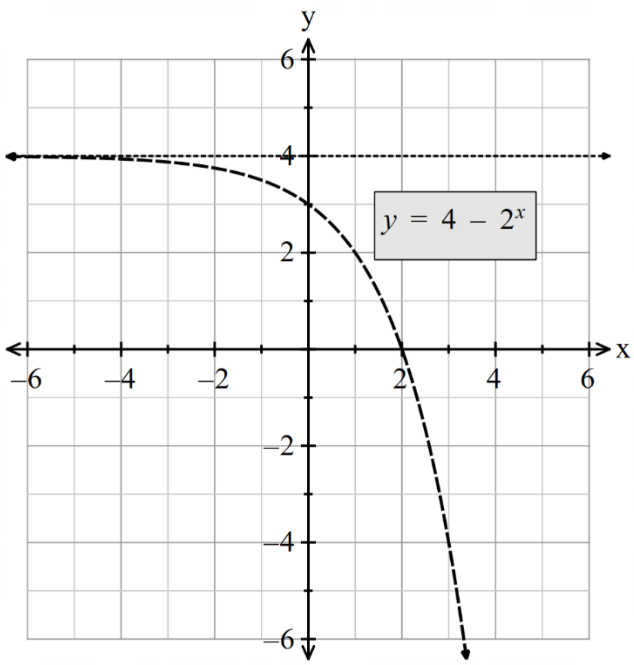
$$\int_0^{\frac{\pi}{6}} \cos^2(3x)\,dx$$

(c) Find the coefficient of x^5 in the binomial expansion of $(2 - 3x)^9$. **2**

(d) Given $\underset{\sim}{u} = (\sqrt{5} - \sqrt{3})\underset{\sim}{i} + \frac{1}{4}\underset{\sim}{j}$ and $\underset{\sim}{v} = (\sqrt{5} + \sqrt{3})\underset{\sim}{i} - 8k\underset{\sim}{j}$, find k such that $\underset{\sim}{u}$ is perpendicular to $\underset{\sim}{v}$. **2**

Continue Question 11 on page 9

(e) Given the graph $y = 4 - 2^x$, sketch $y^2 = 4 - 2^x$. **3**



(f) Solve **3**

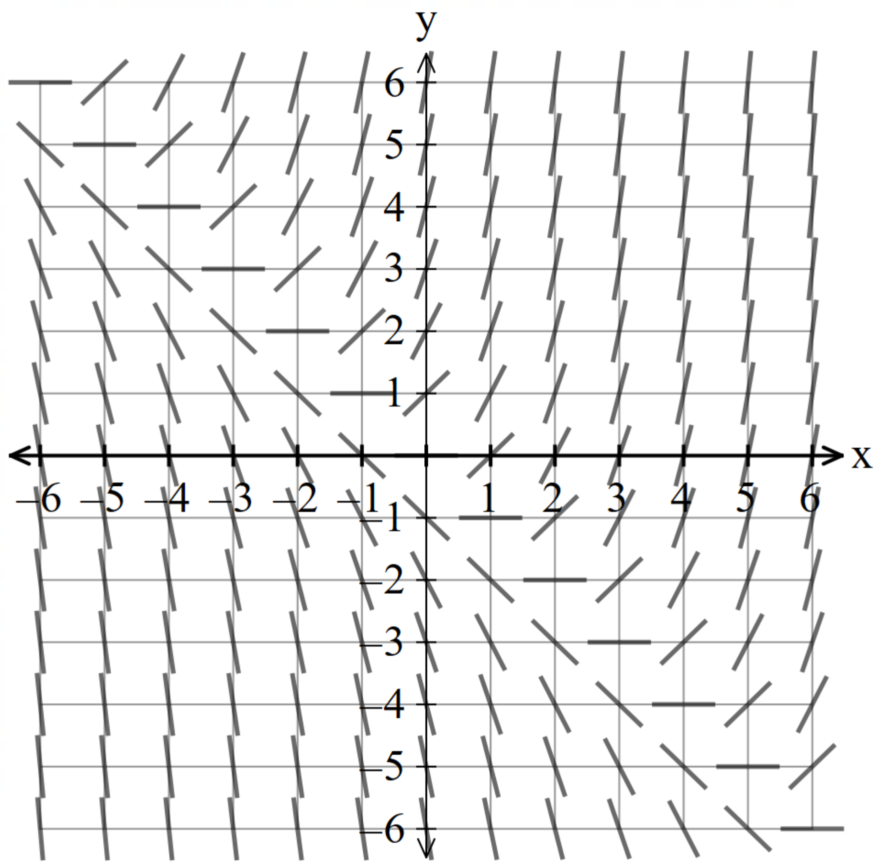
$$\frac{x^2 - 4x + 3}{x + 2} \leq 0$$

(g) Solve the equation $\sin x - \sqrt{3}\cos x = 1$, for $0 \leq x \leq 2\pi$ **3**

End of Question 11

Question 12 (15 marks) Answer each question in a NEW booklet.

(a) On the template on page 17, draw the solution to the given direction field passing through the point $(3, 3)$. **1**



(b) Sand is dropping onto a horizontal floor at the constant rate of 4 cm^3/s and forms a pile. **2**

The formula between volume, $V\ cm^3$ and height $h\ cm$ is shown below

$$V = -8 + \sqrt{h^4 + 64}.$$

Find the rate at which the height of the pile is increasing when the height of the pile has reached 2 cm.

Question 12 continue on page 11

(c) Prove by mathematical induction that, for all integers $n \geq 1$, **3**

$$1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + \ \ldots + n \times 3^n = \frac{3}{4}[(2n-1)3^n + 1]$$

(d) Find the equation of the tangent to the curve $y = \frac{1}{2} - \frac{1}{\pi}\sin^{-1}(2x)$ at the point $P\left(\frac{1}{4}, \frac{1}{3}\right)$. **3**

(e) A thermometer is taken from a room to the outdoors, where the outdoor air temperature is -15°C. After 1 minute the thermometer reads 12°C, and after 5 minutes it reads -1°C.

(i) Show that $T = -15 + (T_0 + 15)e^{kt}$ satisfies the Newton's cooling equation **1**

$$\frac{dT}{dt} = k(T + 15)$$

where T is the thermometer temperature after t minutes and T_0 is the room temperature.

(ii) Find the room temperature, to the nearest °C. **3**

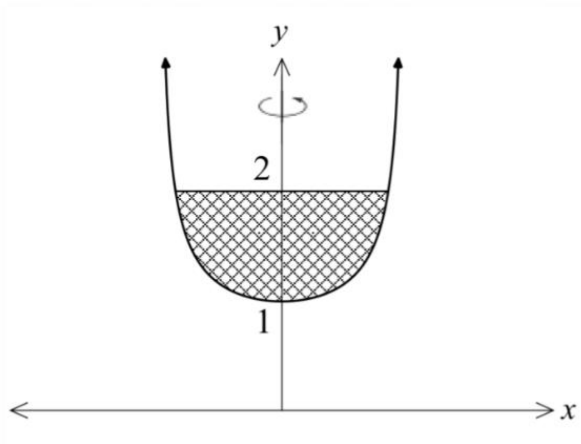
(f) In a factory that manufactures battery cases, the probability of a randomly selected case will fail quality control is 1%. An inspector selects a random batch of 50 cases from the warehouse. **2**

Find the probability that no more than 2 cases from the batch will fail in the test.

End of Question 12

Question 13 (16 marks) Answer each question in a NEW booklet.

(a) A solid is generated by rotating the region bounded by the curve $y = \frac{1}{\sqrt{1-x^2}}$ and the line $y = 2$ about the y axis, as shown in the diagram below. **3**



Find the volume of the solid.

(b) Evaluate $\int_0^4 \frac{3}{\sqrt{8x-x^2}}\, dx$, using the substitution $u = x - 4$. **3**

(c) We call a quadrilateral ABCD a kite if $|AB| = |BC|$ and $|CD| = |DA|$. Let M be the intersection point of AC and BD and it bisects AC. **4**

Use vectors to prove that the diagonals of a kite are perpendicular.

(d) A plane is travelling at 400km/h with a bearing of $40°$. The plane encounters a wind from the south with a speed of 50km/h. **2**

What is the resultant bearing of the plane?

(e) (i) Show that **1**

$$2\cos\frac{x}{2}\sin\frac{5x}{2} = \sin 2x + \sin 3x$$

(ii) By finding a similar expression for $\cos 2x + \cos 3x$, solve the following equation for $x \in [0, 2\pi]$. **3**

$$\sin 2x + \sin 3x + \cos 2x + \cos 3x = 0$$

End of Question 13

Question 14 (13 marks) Answer each question in a NEW booklet.

(a) Steve weighs 80kg and decides to go on a diet of 1600 calories per day. His body automatically uses 850 calories each day to maintain his basic bodily functions.

He spends 15 cal/kg/day doing exercise. 1kg of fat stores 10,000 calories and the process of storing energy in fat is assumed to be 100% efficient such that all 10,000 calories must be used to remove a full 1kg of weight from a person or 5,000 calories need to be used to remove 0.5kg of weight.

(i) Show that the rate of change of Steve's weight (w) over time (t) is given by the equation: **1**

$$\frac{dw}{dt} = \frac{750 - 15w}{10000}$$

(ii) Solve the differential equation to find an expression for Steve's weight. **2**

(iii) If he continues this diet and the exercise regime, will his weight approach an equilibrium in the long term? **1**

(b) NSW has been running a campaign to reduce smoking in young adults. The proportion of smokers in the general population is 21%.

In a survey of 1200 randomly selected young adults, it was found that 226 people smoked regularly.

(i) Find the sample proportion and sample standard deviation. **2**

(ii) NSW are comparing their sample for young adults with the general population. **3**

By finding the probability that no more than 226 people in a sample of 1200 would be regular smokers, determine whether the program has been successful. The program is considered successful if the probability is less than 5%.

Continue Question 14 on page 14

(c) A polynomial function f is defined by **4**

$$f(x) = k(m - x)^3(n - x)^2,$$

where k, m and n are positive constants with $n > m$.

Consider the function $g(x) = f(ax + b)$, where a and b are positive constants. The points $(1, 0)$ and $(2, 0)$ lie on the graph of $y = g(x)$.

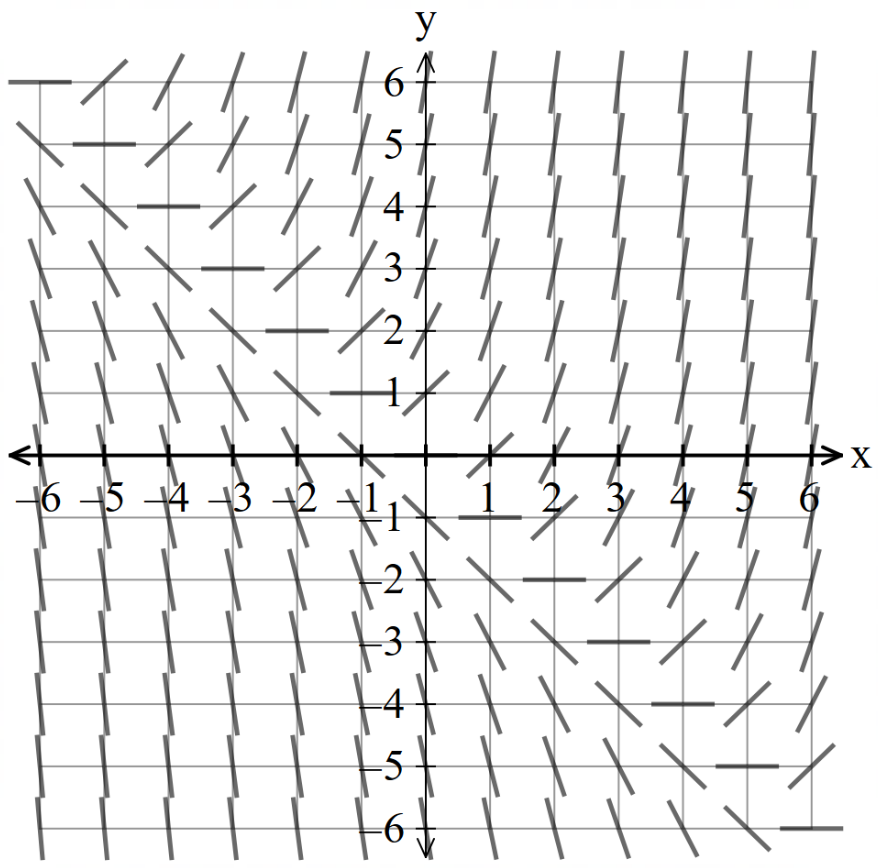
Find the roots of $g'(x)$.

End of paper

Student Number

Question 12

(a) Draw the solution to the given direction field passing through the point $(3, 3)$. **1**



KILLARA HIGH SCHOOL
MATHEMATICS DEPARTMENT

2023 **Year 12 Examination SOLUTIONS**

Mathematics Extension 1

Trial

Date 3rd August, 2023

Q	Marks
MC	/10
11	/16
12	/15
13	/16
14	/13
Total	/70

Comment in relation to the trial marking:

According to HSC marking, if there are two solutions in the question, only first solution is marked.

Question	Answer
1	A
2	D
3	B
4	D
5	D
6	B
7	C
8	C
9	C
10	A

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Answer each question in a NEW booklet.

(a) If $\underset{\sim}{a} = -\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} - 2\underset{\sim}{j}$, find $\underset{\sim}{a} + \underset{\sim}{b}$. **1**

Sample answer

$$\begin{aligned}\underset{\sim}{a} + \underset{\sim}{b} &= -\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{i} - 2\underset{\sim}{j}\\ &= 4\underset{\sim}{i} + \underset{\sim}{j}\end{aligned}$$

Criteria	Marks
• Gives correct answer	1

Question was well done, answer should be in the form of the question.

(b) Evaluate **2**

$$\int_0^{\frac{\pi}{6}} \cos^2(3x)dx$$

Sample answer

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \cos^2(3x)dx &= \frac{1}{2}\int_0^{\frac{\pi}{6}} 1 + \cos 6x\,dx\\ &= \frac{1}{2}\left[x + \frac{\sin 6x}{6}\right]_0^{\frac{\pi}{6}}\\ &= \frac{1}{2}\left[\left(\frac{\pi}{6} + 0\right) - (0 + 0)\right]\\ &= \frac{\pi}{12}\end{aligned}$$

Criteria	Marks
• Provides a correct solution	2
• Attempts to use the identity $\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$ in order to integrate	1

Feedback:
Largest mistake was incorrect use of the identity.

(c) Find the coefficient of x^5 in the binomial expansion of $(2 - 3x)^9$. **2**

Sample answer

x^5 term is given by $\binom{9}{4}(2)^4(-3x)^5$

The coefficient of x^5 is $\binom{9}{4}(2)^4(-3)^5 = -489888$	
Criteria	Marks
• Provides a solution for the correct coefficient	2
• Gives the correct coefficient with incorrect working • Attempts to use the binomial expansion to find the coefficient	1

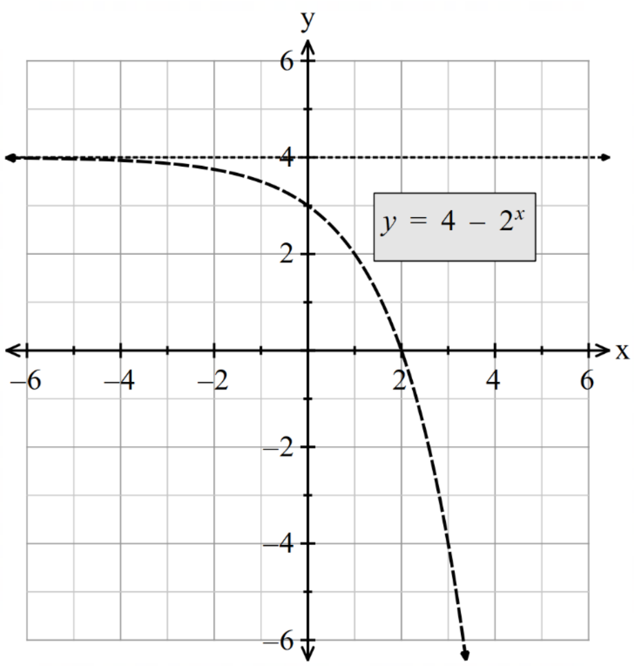
Feedback:
Question was largely well done.

(d) Given $\underset{\sim}{u} = (\sqrt{5} - \sqrt{3})\underset{\sim}{i} + \frac{1}{4}\underset{\sim}{j}$ and $\underset{\sim}{v} = (\sqrt{5} + \sqrt{3})\underset{\sim}{i} - 8k\underset{\sim}{j}$, find k such that $\underset{\sim}{u}$ is perpendicular to $\underset{\sim}{v}$. **2**

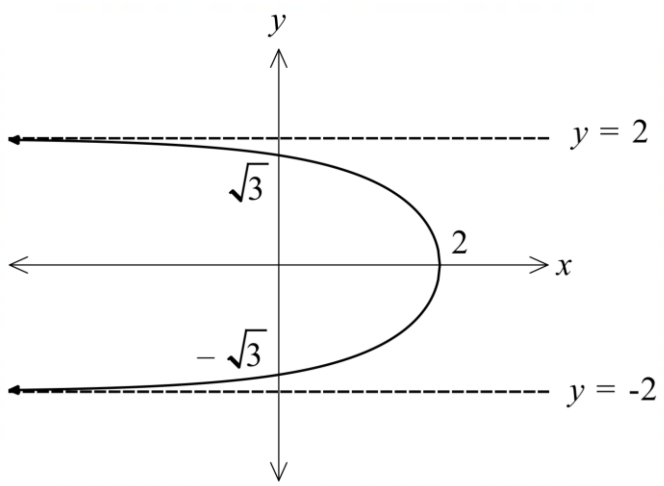
Sample answer Using $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3}) + \frac{1}{4}(-8k) = 0$ $2 - 2k = 0$ $k = 1$	
Criteria	Marks
• Correctly uses the dot product to evaluate k	2
• Recognises that $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ and attempts to use it to find k	1

Feedback:
Question was largely well done

(e) Given the graph $y = 4 - 2^x$, sketch $y^2 = 4 - 2^x$. **3**



Sample answer



Criteria	Marks
• Provides a correct sketch with all intercepts and asymptotes noted	3
• Provides a correct sketch, but either the intercepts or asymptotes are missing or incorrect • Provides a correct $y = \sqrt{4 - 2^x}$ graph without reflecting	2
• Recognises the asymptote of $y = 2$ or the intercept of $\left(0, \sqrt{3}\right)$	1

Feedback:
Many students sketched poorly, with strange kinks. As well, many students did not show all intercepts.

(f) Solve **3**

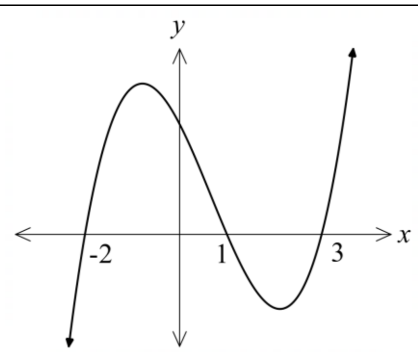
$$\frac{x^2 - 4x + 3}{x + 2} \leq 0$$

Sample answer

$$(x^2 - 4x + 3)(x + 2) \leq 0$$
$$(x - 3)(x - 1)(x + 2) \leq 0$$

$x < -2, 1 \leq x \leq 3$ (from the graph)

Note: $x \neq -2$ because it's not in the domain



Criteria	Marks
• Provides a correct solution	3
• Finds the correct critical values and regions for the inequality, but uses $x \leq -2$	2
•	1

Feedback:
Question was mixed. When students made mistakes, they did so through insufficient working. Too many students multiplied both sides by $(x+2)^2$ and somehow, despite the left being zero, ended up with a non-zero value. This elementary mistake was disappointing. And cost many marks.

(g) Solve the equation $\sin x - \sqrt{3}\cos x = 1$, for $0 \le x \le 2\pi$ **3**

Sample answer

$$\sin x - \sqrt{3}\cos x = R\sin(x-\alpha)$$
$$R = \sqrt{1+3} = 2$$

$$2\sin\alpha = 1 \text{ (coefficient of } \sin x)$$
$$\alpha = \frac{\pi}{6}$$

$$2\sin\left(x-\frac{\pi}{6}\right) = 1$$
$$\sin\left(x-\frac{\pi}{6}\right) = \frac{1}{2}$$
$$x-\frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \dots$$
$$x = \frac{\pi}{3}, \pi$$
For $0 \le x \le 2\pi$

Criteria	Marks
• Provides a correct solution to find the two solutions	3
• Uses the identity $\sin x - \sqrt{3}\cos x = 2\sin\left(x-\frac{\pi}{6}\right)$ and attempts to find solutions	2
•	1

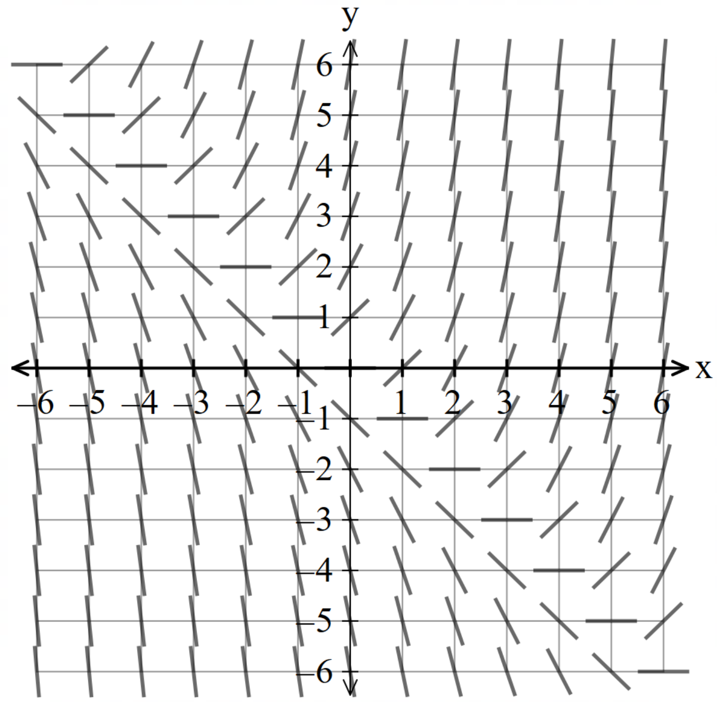
Feedback:

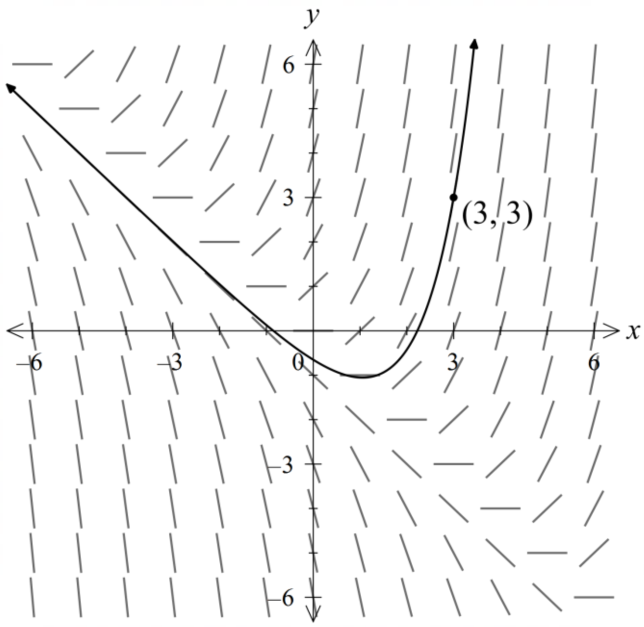
This question was largely well done, mistakes mostly occurring when students missed one of the answers.

End of Question 11

Question 12 (15 marks) Answer each question in a NEW booklet.

(a) On the template on page 17, draw the solution to the given direction field passing through the point $(3,3)$. **1**



Sample answer	
	
Criteria	Marks
• Provides a correct solution, which does not cross over other (the centre of) tangent lines and does not cross the asymptote at $y = -x - 1$	1
Comment: Well done.	

(b) Sand is dropping onto a horizontal floor at the constant rate of 4 cm^3/s and forms a pile. **2**

The formula between volume, $V\ cm^3$ and height $h\ cm$ is shown below

$$V = -8 + \sqrt{h^4 + 64}.$$

Find the rate at which the height of the pile is increasing when the height of the pile has reached 2 cm.

Sample answer

$$\frac{dV}{dt} = 4$$

$$\frac{dV}{dh} = \frac{1}{2}(h^4 + 64)^{-\frac{1}{2}}(4h^3)$$
$$= \frac{2h^3}{\sqrt{h^4 + 64}}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$= \frac{\sqrt{h^4 + 64}}{2h^3} \times 4$$

When $h = 2$, $\frac{dh}{dt} = \frac{\sqrt{64+64}}{64} \times 4$

$= \frac{1}{\sqrt{2}}$ cm/s (or ≈ 0.707 cm/s)

Criteria	Marks
• Correctly evaluates $\frac{dh}{dt}$ when $h = 2$	2
• Finds $\frac{dV}{dh}$ and uses this to attempt to find $\frac{dh}{dt}$	1

Comment:

Common Errors:

(i) Incorrect differentiation (ii) Incorrect substitution

(ii) Incorrect simplification of surds

(c) Prove by mathematical induction that, for all integers $n \geq 1$, **3**

$$1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + \ \dots + n \times 3^n = \frac{3}{4}[(2n-1)3^n + 1]$$

Sample answer

When $n = 1$

LHS $= 1 \times 3^1$ RHS $= \frac{3}{4}[(2-1)3^1 + 1]$

$= 3$ $= \frac{3}{4}(3+1)$

$= 4$

$\therefore$ statement is true when n $= 1$

Assume statement is true when $n = k$, some integer $k \geq 1$, that is

$$1\times 3+2\times 3^2+3\times 3^3+\ \dots\ +k\times 3^k=\frac{3}{4}[(2k-1)3^k+1]$$

Consider $n=k+1$:

RTP: $\quad \text{RHS}=\frac{3}{4}[(2(k+1)-1)3^{k+1}+1]$
$$=\frac{3}{4}[(2k+1)3^{k+1}+1]$$

When $n=k+1$,

$$\begin{aligned}\text{LHS}&=1\times 3+2\times 3^2+3\times 3^3+\ \dots\ +k\times 3^k+(k+1)\times 3^{k+1}\\ &=\frac{3}{4}[(2k-1)3^k+1]+(k+1)\times 3^k\\ &=\frac{3}{4}[2k3^k-3^k+1]+\frac{3}{4}\left[\frac{4}{3}k3^{k+1}+\frac{4}{3}3^{k+1}\right]\\ &=\frac{3}{4}[2k3^k-3^k+1+4k3^k+4(3^k)]\\ &=\frac{3}{4}[(6k+3)3^k+1]\\ &=\frac{3}{4}[(2k+1)3^{k+1}+1]\\ &=\text{RHS}\end{aligned}$$

$\therefore$ statement is true for all integers $n\ge 11$, by mathematical induction.

Criteria *Showing $n=1$ is true *Assuming $n=k$ is true **Proving $p(k)$ true $\rightarrow p(k+1)$ true *Proof structure and statements throughout	Marks
• Has all 5 (*)	3
• Has at least 3 (*)	2
• Has at least 1 (*)	1
Comment: Generally, well done.	

(d) Find the equation of the tangent to the curve $y=\frac{1}{2}-\frac{1}{\pi}\sin^{-1}(2x)$ at the point $P\left(\frac{1}{4},\frac{1}{3}\right)$. **3**

Sample answer

$$y'=-\frac{2}{\pi\sqrt{1-4x^2}}$$

$$y'\left(\frac{1}{4}\right)=-\frac{2}{\pi\sqrt{1-4\left(\frac{1}{4}\right)^2}}$$
$$=-\frac{2}{\pi\sqrt{\frac{3}{4}}}$$
$$=-\frac{4}{\pi\sqrt{3}}$$

Equation of tangent:

$$y = -\frac{4}{\pi\sqrt{3}}x + \frac{1}{\pi\sqrt{3}} + \frac{1}{3}$$	
Criteria	Marks
• Provides a correct solution for the equation	3
• Finds the correct gradient of the curve	2
• Attempts to differentiate $\sin^{-1}(2x)$	1
Comment: General errors: i) Incorrect differentiation of $\sin^{-1} f(x)$, ii) difficulty working with rational coefficients	

(e) A thermometer is taken from a room to the outdoors, where the outdoor air temperature is -15°C. After 1 minute the thermometer reads 12°C, and after 5 minutes it reads -1°C.

(i) Show that $T = -15 + (T_0 + 15)e^{kt}$ satisfies the Newton's cooling equation **1**

$$\frac{dT}{dt} = k(T + 15)$$

where T is the thermometer temperature after t minutes and T_0 is the room temperature.

Sample answer $$\begin{aligned} \text{LHS} &= \frac{dT}{dt} \\ &= k(T_0 + 15)e^{kt} \\ &= k(T + 15) \\ &= \text{RHS} \end{aligned}$$	
Criteria	Marks
• Differentiates to show the result	1

(ii) Find the room temperature, to the nearest °C. **3**

Sample answer $$\begin{aligned} T &= -15 + (T_0 + 15)e^{kt} \\ T(1) = -15 + (T_0 + 15)e^{k} &= 12 \\ (T_0 + 15)e^{k} &= 27 \\ \\ T(5) = -15 + (T_0 + 15)e^{5k} &= -1 \\ (T_0 + 15)e^{5k} &= 14 \\ (T_0 + 15)e^{k} \cdot e^{4k} &= 14 \\ 27e^{k} &= 14 \\ e^{k} &= \frac{14}{27} \end{aligned}$$ $$\begin{aligned} (T_0 + 15)e^{k} &= 27 \\ (T_0 + 15)\frac{14}{27} &= 27 \\ T_0 &= 37.07 \ldots \\ &= 37°\text{C} \end{aligned}$$	
Criteria	Marks

• Provides a correct solution to find T_0	3
• Sets up simultaneous equations to solve for k or e^k	2
Comment: Many students had difficulty solving for k as well as applying log rules to solve for T_0	

(f) In a factory that manufactures battery cases, the probability of a randomly selected case will fail quality control is 1%. An inspector selects a random batch of 50 cases from the warehouse. **2**

Find the probability that no more than 2 cases from the batch will fail in the test.

Sample answer

$$p = 0.99, \qquad q = 0.01$$

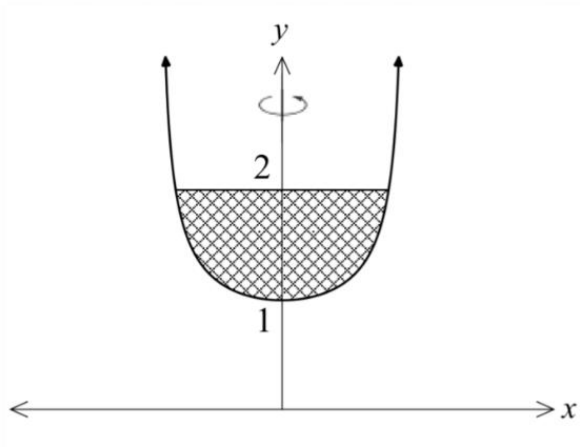
$$P(X \leq 2) = (0.99)^{50} + \binom{50}{1}(0.99)^{49}(0.01) + \binom{50}{2}(0.99)^{48}(0.01)^2$$
$$\approx 98.6\%$$

Criteria	Marks
• Provides a correct solution	2
• Attempts to use binomial probability formulae	1
Comment: Equivalent work using Normal approximation was awarded full marks (need to state and demonstrate $np \geq 5$ *and* $nq \geq 5$ to receive full marks Some students misunderstood the question and evaluated P(x>2) = $1 - P(x \leq 2)$ 0, 1 or 2 are the number of selections from 50, which can be calculated accurately. So, the accurate method is preferred, rather than using the approximation to Normal distribution.	

End of Question 12

Question 13 (16 marks) Answer each question in a NEW booklet.

(a) A solid is generated by rotating the region bounded by the curve $y = \frac{1}{\sqrt{1-x^2}}$ and the line $y = 2$ about the y axis, as shown in the diagram below. **3**



Find the volume of the solid.

Sample Answer:

$$y = \frac{1}{\sqrt{1-x^2}}$$

$$y^2 = \frac{1}{1 - x^2}$$

$$1 - x^2 = \frac{1}{y^2}$$

$$x^2 = 1 - \frac{1}{y^2} = 1 - y^{-2}$$

$$V = \pi \int_1^2 x^2 \, dy$$

$$= \pi \int_1^2 1 - y^{-2} \, dy$$

$$= \pi[y + y^{-1}]_1^2$$

$$= \frac{\pi}{2} \text{ u}^3$$

3 marks:

A correct integration to get $\frac{\pi}{2}$.

2 marks:

Making x^2 the subject and setting up a correct volume equation.

1 mark:

Making x^2 the subject

Feedback:

This was generally done well.

- The most common error was leaving a minus sign in the integral $(= \pi[y - y^{-1}]_1^2)$
- Units were not penalised, but please remember to put cubic units in your answer

(b) Evaluate $\int_0^4 \frac{3}{\sqrt{8x-x^2}}\, dx$, using the substitution $u = x - 4$. **3**

Sample Answer:

$$u = x - 4$$

$$\frac{du}{dx} = 1 \rightarrow du = dx$$

When $x = 4, u = 0$, and when $x = 0, u = -4$

$$\begin{aligned}
\int_0^4 \frac{3}{\sqrt{8x - x^2}}\, dx &= \int_{-4}^{0} \frac{3}{\sqrt{8(u+4) - (u+4)^2}}\, du \\
&= \int_{-4}^{0} \frac{3}{\sqrt{8u + 32 - u^2 - 8u - 16}}\, du \\
&= 3\int_{-4}^{0} \frac{1}{\sqrt{16 - u^2}}\, du \\
&= 3\left[\sin^1\left(\frac{u}{4}\right)\right]_{-4}^{0} \\
&= 3[\sin^{-1} 0 - \sin^{-1}(-1)] \\
&= 3\left[0 - \left(-\frac{\pi}{2}\right)\right] \\
&= \frac{3\pi}{2}
\end{aligned}$$

3 marks:

A correct solution using the given substitution

2 marks:

Changes the integral using the substitution and attempts to use the inverse sine function to integrate

1 mark:

Correctly uses the u-substituion to change the bounds and integral function

Feedback:

This was generally done well. The setting out was quite good.

- The algebra involved with the square root through some people off. Please be careful with fiddly algebra.
- Some students used $\sin^{-1}(-1) = \frac{3\pi}{2}$. Be aware of the range of the inverse sine function.

(c) We call a quadrilateral ABCD a kite if $|AB| = |BC|$ and $|CD| = |DA|$. Let M be the intersection point of AC and BD and it bisects AC. **4**

Use vectors to prove that the diagonals of a kite are perpendicular.

Sample Answer:

$$\begin{aligned}\overrightarrow{BM} &= \overrightarrow{BA} + \overrightarrow{AM}\\ &= \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}\\ &= \overrightarrow{BA} + \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right)\\ &= -\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}\\ &= \frac{1}{2}\overrightarrow{BC} - \frac{1}{2}\overrightarrow{AB}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC}\cdot\overrightarrow{BM} &= \left(\overrightarrow{AB} + \overrightarrow{BC}\right)\cdot\left(\frac{1}{2}\overrightarrow{BC} - \frac{1}{2}\overrightarrow{AB}\right)\\ &= \frac{1}{2}\overrightarrow{AB}\cdot\overrightarrow{BC} + \frac{1}{2}\overrightarrow{BC}\cdot\overrightarrow{BC} - \frac{1}{2}\overrightarrow{AB}\cdot\overrightarrow{AB} - \frac{1}{2}\overrightarrow{AB}\cdot\overrightarrow{BC}\\ &= \frac{1}{2}\left|\overrightarrow{BC}\right|^2 - \frac{1}{2}\left|\overrightarrow{AB}\right|^2\\ &= 0 \qquad \left(\text{as } \left|\overrightarrow{BC}\right| = \left|\overrightarrow{AB}\right|\right)\end{aligned}$$

$\therefore\ \overrightarrow{BM} \perp \overrightarrow{AC}$

Given that BM lies on the diagonal BD, then $\overrightarrow{BD} \perp \overrightarrow{AC}$.

4 marks:

A correct proof with consistent vector notation and a conclusion

3 marks:

A correct proof with inconsistent notation and/or no conclusion given

2 marks:

Correctly finding $\overrightarrow{BM}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}$, and a good attempt was made to find the dot product, which included using the identity $\underset{\sim}{u}\cdot\underset{\sim}{u} = \left|\underset{\sim}{u}\right|^2$

1 mark:

Correctly finding $\overrightarrow{BM}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ (or something equivalent with $\overrightarrow{DM}$)

Feedback:

This was not done well. Most students received 0 marks.

This was a tough question to start because the strategy was not given for you. You will be asked at least two 4/5 mark questions in the HSC, and devising a strategy and executing it correctly is a skill to practise for these.

The question had M referenced in the question, which should have been a hint to start using it. A lot of students did not use M at all, which made it very hard to get 1 mark.

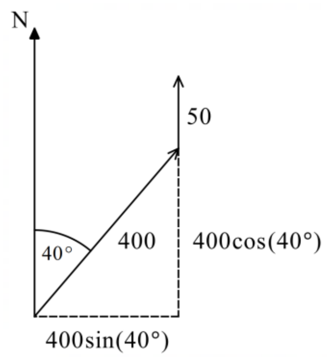
It is possible to prove this relationship without using M, which some students did well.

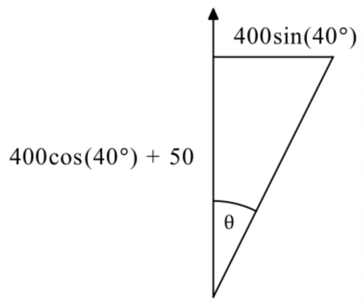
- It was good that knowing $\overrightarrow{AC}\cdot\overrightarrow{BD} = 0$ was needed to prove this results, but stating this was not enough for 1 mark

(d) A plane is travelling at 400km/h with a bearing of $40°$. The plane encounters a wind from the south with a speed of 50km/h. **2**

What is the resultant bearing of the plane?

Sample Answer:





2 marks:

Correctly finds the bearing of the resultant vector

1 mark:

Finds the bearing with the wind in the wrong direction

Or

Finds a bearing with the wrong sin and cos.

Using the diagram above, the resultant vector of the plan with wind is $\underset{\sim}{v} = \binom{400 \sin 40°}{400 \cos 40° + 50}$

For the bearing:

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{400 \sin 40°}{400 \cos 40° + 50}\right) \\ &\approx 036°\text{T}\end{aligned}$$

Feedback:

This question should have been done better. A lot of students received 1 out of 2, and their mistakes were simple and careless. These included:

- Thinking a right angle was $100°$
- Taking the bearing $(40°)$ from the vertical axis rather than from North
- Some mistakes with right angle trigonometry when using sin or cos

Please make sure you leave your answer as a bearing, since that's what the question asked for.

(e) (i) Show that **1**

$$2\cos\frac{x}{2}\sin\frac{5x}{2} = \sin 2x + \sin 3x$$

(e) (i) **Sample Answer:** **1**

$\sin 2x + \sin 3x = sin\left(\frac{5x}{2} - \frac{x}{2}\right) + sin\left(\frac{5x}{2} + \frac{x}{2}\right)$

$= sin\left(\frac{5x}{2}\right)\cos\frac{x}{2} - cos\left(\frac{5x}{2}\right)\sin\frac{x}{2} + sin\left(\frac{5x}{2}\right)\cos\frac{x}{2}$

$+ cos\left(\frac{5x}{2}\right)\sin\frac{x}{2}$

$= 2\,sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right) = LHS$

Feedback:

This was done well across the board.

(ii) By finding a similar expression for $\cos 2x + \cos 3x$, solve the following equation for $x \in [0, 2\pi]$. **3**

$\sin 2x + \sin 3x + \cos 2x + \cos 3x = 0$

(ii) **Sample Answer:**

$\cos 2x + \cos 3x = cos(\frac{5x}{2} - \frac{x}{2}) + cos(\frac{5x}{2} + \frac{x}{2})$

$= \cos\frac{5x}{2}\cos\frac{x}{2} + \sin\frac{5x}{2}\sin\frac{x}{2} + \cos\frac{5x}{2}\cos\frac{x}{2} - \sin\frac{5x}{2}\sin\frac{x}{2}$

$= 2\cos\frac{5x}{2}\cos\frac{x}{2}$

Solve:

$2\,sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right) + 2\cos\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right) = 0$

$2\cos\left(\frac{x}{2}\right)\left(sin\left(\frac{5x}{2}\right) + \cos\frac{5x}{2}\right) = 0$

$\cos\frac{x}{2} = 0$

$\frac{x}{2} = \frac{\pi}{2}$, where $\frac{x}{2} \in [0, \pi]$

$\therefore x = \pi$

OR

5 markers:

* recognising $\cos 2x + \cos 3x = 2\cos\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right)$

* Setting up $2\cos\left(\frac{x}{2}\right)\left(\sin\left(\frac{5x}{2}\right) + \cos\left(\frac{5x}{2}\right)\right) = 0$

* solving $\cos\left(\frac{x}{2}\right) = 0$

* solving $\sin\left(\frac{5x}{2}\right) + \cos\left(\frac{5x}{2}\right) = 0$

* covering all solutions in the domain

$sin\left(\frac{5x}{2}\right)+\ \cos\frac{5x}{2}=0$

$\tan\frac{5x}{2}\ =-1,\ \ \frac{5x}{2}\in[0,5\pi]$

$\frac{5x}{2}=\pi-\frac{\pi}{4},2\pi-\frac{\pi}{4},2\pi+\frac{3\pi}{4},2\pi+\frac{7\pi}{4},4\pi+\frac{3\pi}{4}$

$\frac{5x}{2}=\frac{3\pi}{4},\frac{7\pi}{,4},\frac{11\pi}{4},\frac{15\pi}{4},\frac{19\pi}{4}$

$x=\frac{3\pi}{4}\times\frac{2}{5},\frac{7\pi}{,4}\times\frac{2}{5},\frac{11\pi}{4}\times\frac{2}{5},\frac{15\pi}{4}\times\frac{2}{5},\frac{19\pi}{4}\times\frac{2}{5}$

$$x=\frac{3\pi}{10},\frac{7\pi}{,10},\frac{11\pi}{10},\frac{3\pi}{2},\frac{19\pi}{10}$$

3 marks: for five *

2 marks: at least 3 *

1 mark: at least 1 *

So the solutions are $x=\pi,\ \frac{3\pi}{10},\frac{7\pi}{,10},\frac{11\pi}{10},\frac{3\pi}{2},\frac{19\pi}{10}$

Feedback:

This was done reasonably well. Most students correctly identified $\cos 2x+\cos 3x=2\cos\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right)$ and then recognised that they had to solve

$$2\cos\left(\frac{x}{2}\right)\left(\sin\left(\frac{5x}{2}\right)+\cos\left(\frac{5x}{2}\right)\right)=0$$

The biggest mistakes were:

- Forgetting $\cos\left(\frac{x}{2}\right)=0$ needs to be solved
- Forgetting ALL solutions in the domain for $\tan\left(\frac{5x}{2}\right)=-1$ (many students stopped with the first two)

End of Question 13

Question 14 (13 marks) Answer each question in a NEW booklet.

(a) Steve weighs 80kg and decides to go on a diet of 1600 calories per day. His body automatically uses 850 calories each day to maintain his basic bodily functions.

He spends 15 cal/kg/day doing exercise. 1kg of fat stores 10,000 calories and the process of storing energy in fat is assumed to be 100% efficient such that all 10,000 calories must be used to remove a full 1kg of weight from a person or 5,000 calories need to be used to remove 0.5kg of weight.

(i) Show that the rate of change of Steve's weight (w) over time (t) is given by the equation: **1**

$$\frac{dw}{dt} = \frac{750 - 15w}{10000}$$

(ii) Solve the differential equation to find an expression for Steve's weight. **2**

(iii) If he continues this diet and the exercise regime, will his weight approach an equilibrium in the long term? **1**

(a) **Sample Answer:**

(i) Input Calories per a day = 1600 – 850 **1**

= 750

Output Calories per a day = 15w

Net Calories per a day = 750 – 15w

$$\therefore \frac{dw}{dt} = \frac{750 - 15w}{10000}.$$

Feedback:

Quite few students could not interpret the weight rate equation.

(ii) $\frac{dt}{dw} = \frac{10000}{750 - 15w}$

1 mark: correct answer with working for finding constant C.

$\int dt = \int \frac{10000}{750 - 15w} dw$

$t = -\frac{10000}{15} \ln|750 - 15w| + C$

1 mark: correct answer with working for finding the expression of w in terms of t.

When t = 0, w = 80kg

$0 = -\frac{2000}{3} \ln|750 - 15 \times 80| + C$

$= -\frac{2000}{3} \ln|-450| + C$

$\therefore C = \frac{2000}{3}\ln 450.$

$t - \frac{2000}{3}\ln 450 = -\frac{2000}{3}\ln|750 - 15w|$

$-\frac{3}{2000}t + \ln 450 = \ln|750 - 15w|$

$e^{-\frac{3}{2000}t + \ln 450} = 15w - 750$

$15w = 750 + 450e^{-\frac{3}{2000}t}$

$\therefore w = 50 + 30e^{-\frac{3}{2000}t}$

Feedback:

Quite a lot of students integrated as a log function, but left the constant A or C as the final answer.

(iii) If he continues this diet and the exercise regime, he will reach to 50kg. **1**

Feedback:

Most students got correct conclusions based on ii).

(b) NSW has been running a campaign to reduce smoking in young adults. The proportion of smokers in the general population is 21%.

In a survey of 1200 randomly selected young adults, it was found that 226 people smoked regularly.

(i) Find the sample proportion and sample standard deviation. **2**

(ii) NSW are comparing their sample for young adults with the general population. **3**

By finding the probability that no more than 226 people in a sample of 1200 would be regular smokers, determine whether the program has been successful. The program is considered successful if the probability is less than 5%.

(b) **Sample Answer:**

(i) **Method 1:**

1 mark: correct sample proportion.

$$\hat{p} = \frac{226}{1200} = \frac{113}{600}$$

$\mathrm{Var}(\hat{p}) = \frac{pq}{n}$

$= \frac{0.21\times(1-0.21)}{1200}$

$= 0.00013825$

$\sigma \approx 0.01176$

1 mark: correct standard deviation with working.

Method 2:

$\sigma = \sqrt{npq}$

$= \sqrt{1200\ \times\ 0.21\ \times\ 0.79}$

≈ 14.10957

(ii) **Method 1:**

$P(\hat{p} \leq \frac{226}{1200})$, $\quad z = \frac{\hat{p}-p}{\sigma} = \frac{\frac{113}{600}-0.21}{0.01176} = -1.8424$

$= P(z \leq\ -1.8424)$

$= 1 - P(z \leq 1.8424)$

$= 1 - 0.96731$

$= 0.03269$

There is only 3%, therefore it is successful.

1 mark: correct z – score with working.

1 mark: correct probability with working.

1 mark: correct conclusion.

Method 2:

$E(X) = np = 1200\times0.21 = 252$

$z = \frac{226-252}{\sigma} = \frac{226-252}{14.10957} = -1.8427$

$P(X \leq\ 226)$,

$= P(z \leq\ -1.8427)$

$= 1 - P(z \leq 1.8427)$

=

There is only 3%, therefore it is successful.

Feedback:

Most students who attempted Q14, able to apply either method 1 or 2.

Continue Question 14 on page 13

(c) A polynomial function f is defined by **4**

$$f(x) = k(m - x)^3(n - x)^2,$$

where k, m and n are positive constants with $n > m$.

Consider the function $g(x) = f(ax + b)$, where a and b are positive constants. The points $(1, 0)$ and $(2, 0)$ lie on the graph of $y = g(x)$.

Find the roots of $g'(x)$.

(c) **Sample Answer 1:** **4**

$g(x) = k(m - ax - b)^3(n - ax - b)^2$

$$\begin{aligned} g'(x) &= 3k(m - ax - b)^2(-a)(n - ax - b)^2 + 2k(m - ax - b)^3(-a)(n - ax - b) \\ &= -ak(m - ax - b)^2(n - ax - b)(3n - 3ax - 3b + 2m - 2ax - 2b) \\ &= -ak(m - ax - b)^2(n - ax - b)(3n - 5ax - 5b + 2m) \end{aligned}$$

$3n - 5ax - 5b + 2m = 0$

$m - ax - b = 0, \quad n - ax - b = 0$

$x = \frac{m-b}{a}, \ x = \frac{n-b}{a}$

Because the points $(1, 0)$ and $(2, 0)$ lie on the graph of $y = g(x)$, and $n > m$

$1 = \frac{m-b}{a}, \ 2 = \frac{n-b}{a}$

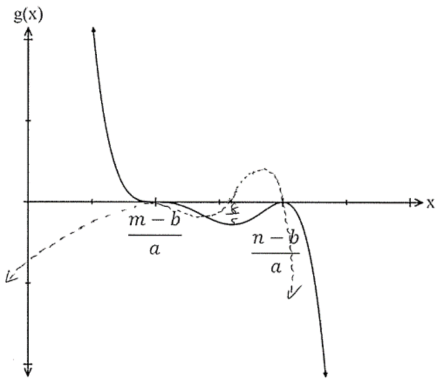
$2m - 2b = n - b$

$b = 2m - n$

$3n - 5ax - 5b + 2m = 0$

$3n - 5x(m - b) - 5b + 2m = 0$

$3n - 5x(m - 2m + n) - 5(2m - n) + 2m = 0$



$8n - 8m = 5(n - m)x$

$\therefore x = \frac{8}{5}$

$\therefore$ Roots of $g'(x)$ are $(1, 0)$, $(\frac{8}{5}, 0)$ and $(2, 0)$.

Marking Scheme:

1 mark: correct g(x).
1 mark: correct g'(x) factorisation.
1 mark: two correct equations in terms of m, n, a, & b when substitute two given points.
1 mark: substitution with a & b in terms of m & n.

Sample Answer 2:

$f(x) = -k(x - m)^3(x - n)^2$

$f(ax + b) = -k(ax+b-m)^3(ax+b-n)^2$

$$= -k\left(a\left(x+\frac{b-m}{a}\right)\right)^3\left(a\left(x+\frac{b-n}{a}\right)\right)^2$$
$$= -k\left(a\left(x-\frac{m-b}{a}\right)\right)^3\left(a\left(x-\frac{n-b}{a}\right)\right)^2$$

Translate: $(x-m) \to \left(x-\frac{m-b}{a}\right)$, $(x-n) \to \left(x-\frac{n-b}{a}\right)$.
$\left(x-\frac{m-b}{a}\right) = 0$ and $\left(x-\frac{n-b}{a}\right) = 0$.
Because the points $(1,0)$ and $(2,0)$ lie on the graph of $y = g(x)$, and $n > m$

$\frac{m-b}{a} = 1$ or $\frac{n-b}{a} = 2$.

$\boldsymbol{m - b = a}$, $\frac{n-b}{2} = a$

$m - b = \frac{n-b}{2}$

$2m - 2b = n - b$

$\boldsymbol{b = 2m - n}$

$f'(x) = -3k(x-m)^2(x-n)^2 - 2k(x-m)^3(x-n)$

$= -k(x-m)^2(x-n)(3x-3n+2x-2m)$

$= -k(x-m)^2(x-n)(5x-3n-2m)$

$5x - 3n - 2m = 0$,

Translate: $x = \frac{2m+3n}{5} \rightarrow \frac{2m+3n}{5a} - \frac{b}{a} = \frac{2m+3n-5b}{5a}$

$$= \frac{2m+3n-5(\boldsymbol{2m-n})}{5(\boldsymbol{n-m})},$$

$$= \frac{2m+3n-10m+5\boldsymbol{n}}{5(\boldsymbol{n-m})}$$

$$= \frac{8(n-m)}{5(n-m)}$$

$$= \frac{8}{5}$$

$\therefore$ Roots of $g'(x)$ are $(1,0)$, $(\frac{8}{5}, 0)$ and $(2,0)$.

Marking Scheme:

1 mark: correct f'(x) factorisation.
1 mark: correct implement two given points to find b in terms of m & n.
1 mark: correct transformations from f'(x) factors, example $(x-m) \rightarrow \left(x-\frac{m-b}{a}\right)$.
1 mark: correct finding the third root with working.

Feedback:

Most students who attempt part C) can get at least 1 mark, that is showing correct g(x). Quite a few students did not know how to apply correctly with the given points (1,0) and (2,0) and the condition $n > m$.

End of paper